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## **Precise determination of the strong coupling constant at NNLO in QCD from the three-jet rate in electron–positron annihilation at LEP**

Dissertori, G ; Gehrmann-De Ridder, A ; Gehrmann, T ; Glover, E W N ; Heinrich, G ; Stenzel, H

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# Precise determination of the strong coupling constant at NNLO in QCD from the three-jet rate in electron–positron annihilation at LEP

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We present the first determination of the strong coupling constant from the three-jet rate in  $e^+e^-$  annihilation at LEP, based on a next-to-next-to-leading order (NNLO) perturbative QCD prediction. More precisely, we extract  $\alpha_s(M_Z)$  by fitting perturbative QCD predictions at  $\mathcal{O}(\alpha_s^3)$  to data from the ALEPH experiment at LEP. Over a large range of the jet-resolution parameter  $y_{\text{cut}}$  this observable is characterised by small non-perturbative corrections and an excellent stability under renormalisation scale variation. For the three-jet rate at  $\ln y_{\text{cut}} = -3.9$  we find  $\alpha_s(M_Z) = 0.1175 \pm 0.0020$  (exp)  $\pm 0.0015$  (theo), which is among the most precise determinations from jet observables at LEP.

PACS numbers: 12.38.Bx, 13.66.Bc, 13.66.Jn, 13.87.-a

Jet observables in electron–positron annihilation play an outstanding role in studying the dynamics of the strong interactions, described by the theory of Quantum Chromodynamics (QCD, [1]). In particular, jet rates and related event-shape observables have been extensively used for the determination of the QCD coupling constant  $\alpha_s$  (see [2, 3] for a review), mostly based on data obtained at the  $e^+e^-$  colliders PETRA, LEP and SLC at centre-of-mass energies from 14 to 209 GeV. Jets are defined using a jet algorithm, which describes how to recombine the momenta of all particles in an event to form the jets. A jet algorithm consists of two ingredients: a distance measure and a recombination procedure. The distance measure is computed for each pair of momenta to select the pair with the smallest separation. If the separation is below a pre-defined resolution parameter  $y_{\text{cut}}$ , the pair of momenta are combined according to the recombination procedure into a joint momentum. The JADE algorithm [5] uses the pair invariant mass as distance measure. Several improved jet algorithms have been proposed for  $e^+e^-$  collisions: Durham [6], Geneva [7] and Cambridge [8]. The Durham algorithm has been the most widely used by experiments at LEP [9, 10, 11, 12] and SLD [13], as well as in the reanalysis of earlier data at lower energies from JADE [14].

The Durham jet algorithm clusters particles into jets by computing the distance measure

$$y_{ij,D} = \frac{2 \min(E_i^2, E_j^2)(1 - \cos \theta_{ij})}{E_{\text{vis}}^2} \quad (1)$$

for each pair  $(i, j)$  of particles,  $E_{\text{vis}}$  denotes the energy sum of all particles in the final state. The pair with the lowest  $y_{ij,D}$  is replaced by a pseudo-particle whose four-momentum is given by the sum of the four-momenta of particles  $i$  and  $j$  (' $E$ ' recombination scheme). This procedure is repeated as long as pairs with invariant mass below the predefined resolution parameter  $y_{ij,D} < y_{\text{cut}}$

are found. Once the clustering is terminated, the remaining (pseudo-)particles are the jets. It is evident that for a large value of  $y_{\text{cut}}$  all particles are ultimately clustered into only two jets, while higher jet multiplicities become more and more frequent as  $y_{\text{cut}}$  is lowered. In experimental jet measurements, one therefore studies the jet rates (jet cross sections normalised to the total hadronic cross section) as function of the jet-resolution parameter  $y_{\text{cut}}$ .

The theoretical prediction of jet cross sections is made within perturbative QCD, where the same jet algorithm is applied to the momenta of final state partons. The QCD description of jet production is either based on a fixed-order calculation or a parton shower. The fixed order approach uses exact parton-level matrix elements including higher order corrections where available and/or analytical resummation of large logarithmic corrections for a given jet multiplicity. On the other hand, the parton shower starts with the leading-order matrix element for two-jet production and generates higher multiplicities in an iterative manner, thereby accounting only for the leading logarithmic terms from parton-level processes with higher multiplicity. In multi-purpose event generator programs [15, 16, 17], such parton showers are complemented by phenomenological models which describe the transition from partons to hadrons. These programs provide a satisfactory description of multi-jet production rates but, since they generally contain many tunable phenomenological parameters, their predictive power is very limited.

Until recently, fixed-order calculations were available up to next-to-next-to-leading order (NNLO) for two jets [18, 19, 20] and up to next-to-leading order (NLO) for three [21, 22, 23] and four jets [24, 25, 26, 27]. For five and more jets, only leading order calculations are available [28, 29, 30]. For jets involving massive quarks, NLO results are available for three-jet final states [31]. The recent calculations of the  $\alpha_s^3$  corrections (NNLO) for

three-jet production [32, 33, 34, 35] have already led to precise  $\alpha_s$  determinations [36, 37, 38, 39, 40], using event-shape observables measured by ALEPH and JADE. However, some of the event-shape variables still suffer from a poor convergence of the perturbative expansion even at NNLO. Furthermore, the usage of event generators, which have been tuned to LEP data, for the determination of the hadronisation corrections may lead to a bias in the  $\alpha_s$  measurements for some of the event shapes [40]. A comparison of different variables showed that jet broadening variables are most affected by missing higher orders and a potential hadronisation bias, while the differential two-jet rate  $Y_3$  is most robust against these effects, and strongly motivates the present study of the three-jet rate.

In this letter we describe a determination of the strong coupling constant from the three-jet rate measured by ALEPH [41] at LEP. We use the NNLO predictions as presented in [33]. There it was shown that: (i) For large values of  $y_{\text{cut}}$ ,  $y_{\text{cut}} > 10^{-2}$ , the NNLO corrections turn out to be very small, while they become substantial for medium and low values of  $y_{\text{cut}}$ ; (ii) The maximum of the jet rate is shifted towards higher values of  $y_{\text{cut}}$  compared to NLO and is in better agreement with the experimental observations; (iii) The theoretical uncertainty is lowered considerably compared to NLO, especially in the region  $10^{-1} > y_{\text{cut}} > 10^{-2}$  relevant for precision phenomenology where the theory error is below two per-cent relative uncertainty; (iv) Finally, in this  $y_{\text{cut}}$  region the parton level predictions at NNLO are already very close to the experimental measurements, indicating the need for only small hadronisation corrections.

These findings motivate a dedicated analysis of the three-jet rate, leading to a precise measurement of  $\alpha_s$ . Our analysis closely follows the procedure described in [36, 40]. The ALEPH data [41] were obtained at centre-of-mass energies between  $Q = 91.2$  GeV and  $Q = 206$  GeV, and are based on the reconstructed momenta and energies of charged and neutral particles. The measurements have been corrected for detector effects, i.e. the final distributions correspond to the so-called particle (or hadron) level, and for initial state photonic radiation. In the simulation of the detector response to particles, a bias is introduced by the choice of the physics event generator. This leads to a systematic uncertainty on the three-jet rate of about 1.5% for the relevant  $y_{\text{cut}}$  range. Further experimental systematic effects are estimated by a variation of the track- and event-selection cuts as advocated in [41], giving an additional small systematic uncertainty of about 1%. In our previous analyses of event shapes [36, 40] we included also data above the  $Z$  peak, but their statistical uncertainties are too large for a precise determination of  $\alpha_s$  from the three-jet rate.

We construct the perturbative expansion up to  $\mathcal{O}(\alpha_s^3)$  as described in [40], with the coefficients obtained from [33]. These are valid for massless quarks. We take into account bottom mass effects up to NLO [31], for a pole b-quark mass of  $M_b = 4.5$  GeV. The latter is varied by  $\pm 0.5$  GeV in order to estimate the impact of the b-quark

mass uncertainty on the value of the strong coupling. For the normalisation to the total hadronic cross section  $\sigma_{\text{had}}$  we follow the procedure adopted in [40], which is based on a N<sup>3</sup>LO calculation ( $\mathcal{O}(\alpha_s^3)$  in QCD) for  $\sigma_{\text{had}}$  [42], including mass corrections for the b-quark up to  $\mathcal{O}(\alpha_s)$  and the leading mass terms to  $\mathcal{O}(\alpha_s^2)$ . Weak corrections to the three-jet rate were computed very recently [43]. They are at the one per-mille level for  $Q = M_Z$  and are neglected here.

The nominal value for the renormalisation scale  $x_\mu = \mu/Q$  is unity. It is varied between  $0.5 < x_\mu < 2$  in order to assess the systematic uncertainty related to yet unknown higher order corrections. No attempt is made to combine the NNLO predictions with resummation calculations. At present, the resummation of the three-jet rate [6] is only fully consistent at leading logarithmic level [44], and resummation effects only become numerically relevant over fixed-order NNLO for  $\ln y_{\text{cut}} \lesssim -4.5$  (as can be seen from the  $Y_3$  transition parameter distribution [45]), which is below our region of interest.

The perturbative QCD prediction is corrected for hadronisation and resonance decays by means of a bin-by-bin correction computed with the PYTHIA [15], HERWIG [16] and ARIADNE [17] Monte Carlo generators, all tuned to global hadronic observables at  $M_Z$  [46]. The parton level is defined by the quarks and gluons present at the end of the parton shower in PYTHIA and HERWIG and the partons resulting from the colour dipole radiation in ARIADNE. Our central values for the strong coupling constant are obtained with hadronisation corrections from PYTHIA, which are at the level of 5%. We define the systematic uncertainty on  $\alpha_s(M_Z)$  due to these hadronisation corrections as the biggest deviation observed when using any of the other generators. Motivated by the observations in [40], we verified that the shapes of the Monte Carlo parton level predictions are in fair agreement with those at NNLO, for reasonable choices of the strong coupling. Furthermore, the ratios of these predictions are relatively flat over the relevant  $y_{\text{cut}}$  range, giving further confidence in the reliability of the hadronisation corrections.

The corrected ALEPH measurements for the three-jet rate are compared to the theoretical calculation at particle level. Values for  $\alpha_s(M_Z)$  are obtained by a least-squares fit, performed separately for each  $y_{\text{cut}}$  value in the range listed in Table I (for the data at the  $Z$  peak), together with the uncertainties as described above. These results are also displayed in Fig. 1. We observe a nice stability of the results, within their total uncertainties, down to resolution parameters of  $\ln y_{\text{cut}} \approx -4.5$ . Beyond that value we find a fall-off of  $\alpha_s(M_Z)$ , most likely related to the onset of large logarithmic corrections from higher perturbative orders, which are not accounted for in our perturbative prediction.

As final result we quote our measurement for the resolution parameter  $\ln y_{\text{cut}} = -3.9$  (corresponding to  $y_{\text{cut}} = 0.02$ ), which represents an optimal compromise between minimal systematic uncertainty and stability.

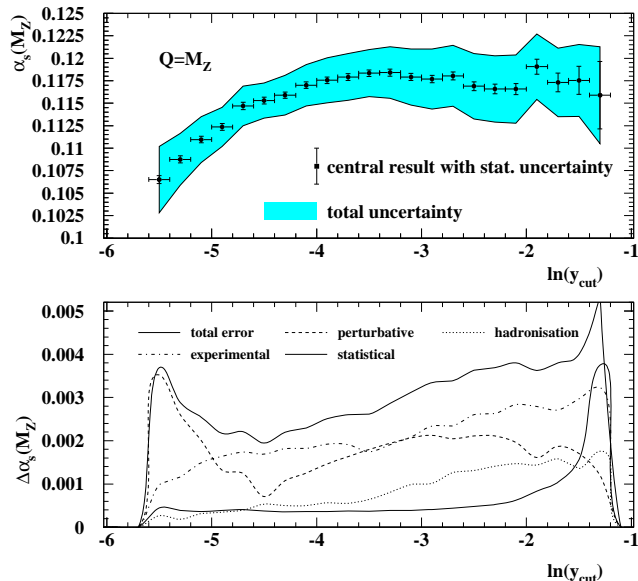


FIG. 1: Determinations of  $\alpha_s(M_Z)$  from the three-jet rate, measured by ALEPH at the  $Z$  peak, for several values of the jet-resolution parameter  $y_{\text{cut}}$ . The error bars show the statistical uncertainty, whereas the shaded band indicates the total error, including the systematic uncertainty. The various contributions to the latter are displayed in the lower plot.

We find

$$\alpha_s(M_Z) = 0.1175 \pm 0.0020 (\text{exp}) \pm 0.0015 (\text{theo}) ,$$

where the first uncertainty includes (in quadrature) the contributions from statistics, detector corrections and experimental selection cuts, and the second error is the quadratic sum of b-quark mass and renormalisation scale

uncertainties (cf. Table I). As expected, the theoretical uncertainties are smaller than those obtained from fits of event-shape distributions, and even smaller than the experimental error, which is dominated by the model-dependence of the detector corrections. Our result is among the most precise determinations of the strong coupling constant at LEP from jet observables, and is in excellent agreement with the latest world average value [3]. It is worth noting that we performed similar measurements also for the LEP2 energies between 133 and 206 GeV, where we find consistent values for  $\alpha_s(M_Z)$ , but with considerably larger statistical uncertainties.

In this letter we reported on the first determination of the strong coupling constant from the three-jet rate in  $e^+e^-$  annihilation at LEP, based on a NNLO perturbative QCD prediction. We find a precise value of  $\alpha_s(M_Z)$  with an uncertainty of 2%, consistent with the world average. This verifies the expectations that the three-jet rate is an excellent observable for this kind of analysis, thanks to the good behaviour of its perturbative and non-perturbative contributions over a sizable range of jet-resolution parameters.

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- [1] H. Fritzsch, M. Gell-Mann and H. Leutwyler, Phys. Lett. B **47** (1973) 365; D.J. Gross and F. Wilczek, Phys. Rev. D **8** (1973) 3633; H.D. Politzer, Phys. Rept. **14** (1974) 129.
  - [2] O. Biebel, Phys. Rept. **340** (2001) 165; S. Kluth, Rept. Prog. Phys. **69** (2006) 1771.
  - [3] S. Bethke, Prog. Part. Nucl. Phys. **58** (2007) 351; arXiv:0908.1135.
  - [4] R. Barate *et al.* [ALEPH Collaboration], Z. Phys. C **76** (1997) 1; R. Akers *et al.* [OPAL Collaboration], Z. Phys. C **65** (1995) 367; P. Abreu *et al.* [DELPHI Collaboration], Z. Phys. C **59** (1993) 357.
  - [5] W. Bartel *et al.* [JADE Collaboration], Z. Phys. C **33** (1986) 23.
  - [6] S. Catani, Y.L. Dokshitzer, M. Olsson, G. Turnock and B.R. Webber, Phys. Lett. B **269** (1991) 432; N. Brown and W.J. Stirling, Phys. Lett. B **252** (1990) 657; Z. Phys. C **53** (1992) 629; W.J. Stirling *et al.*, Proceedings of the Durham Workshop, J. Phys. **G17** (1991) 1567.
  - [7] S. Bethke, Z. Kunszt, D.E. Soper and W.J. Stirling, Nucl. Phys. B **370** (1992) 310; **523** (1998) 681 (E).
  - [8] Y.L. Dokshitzer, G.D. Leder, S. Moretti and B.R. Webber, JHEP **9708** (1997) 001.
  - [9] A. Heister *et al.* [ALEPH Collaboration], Eur. Phys. J. C **35** (2004) 457.
  - [10] G. Abbiendi *et al.* [OPAL Collaboration], Eur. Phys. J. C **40** (2005) 287.
  - [11] P. Achard *et al.* [L3 Collaboration], Phys. Rept. **399** (2004) 71.
  - [12] J. Abdallah *et al.* [DELPHI Collaboration], Eur. Phys. J. C **29** (2003) 285.
  - [13] K. Abe *et al.* [SLD Collaboration], Phys. Rev. D **51** (1995) 962.
  - [14] P. Pfeifenschneider *et al.* [JADE collaboration], Eur. Phys. J. C **17** (2000) 19.
  - [15] T. Sjostrand, P. Eden, C. Friberg, L. Lönnblad, G. Miu, S. Mrenna and E. Norrbin, Comput. Phys. Commun. **135** (2001) 238.

$\ln(y_{\text{cut}})$	$\alpha_s(M_Z)$	stat.	det.	exp.	had.	mass	pert.	total
-5.5	0.1065	0.0004	0.0008	0.0006	0.0003	0.0004	0.0035	0.0037
-5.3	0.1088	0.0004	0.0010	0.0006	0.0002	0.0004	0.0026	0.0029
-5.1	0.1110	0.0004	0.0013	0.0008	0.0003	0.0004	0.0020	0.0025
-4.9	0.1124	0.0004	0.0015	0.0007	0.0003	0.0003	0.0013	0.0022
-4.7	0.1147	0.0004	0.0015	0.0008	0.0004	0.0003	0.0012	0.0022
-4.5	0.1153	0.0004	0.0015	0.0008	0.0005	0.0003	0.0006	0.0019
-4.3	0.1159	0.0004	0.0016	0.0009	0.0005	0.0003	0.0010	0.0022
-4.1	0.1170	0.0004	0.0016	0.0009	0.0005	0.0003	0.0012	0.0023
-3.9	0.1175	0.0004	0.0016	0.0011	0.0006	0.0002	0.0014	0.0025
-3.7	0.1179	0.0004	0.0016	0.0011	0.0006	0.0002	0.0016	0.0026
-3.5	0.1183	0.0004	0.0015	0.0009	0.0006	0.0002	0.0018	0.0026
-3.3	0.1184	0.0004	0.0015	0.0011	0.0008	0.0002	0.0019	0.0029
-3.1	0.1179	0.0004	0.0016	0.0013	0.0010	0.0002	0.0021	0.0031
-2.9	0.1177	0.0004	0.0019	0.0013	0.0010	0.0002	0.0021	0.0033
-2.7	0.1180	0.0004	0.0020	0.0013	0.0013	0.0001	0.0020	0.0034
-2.5	0.1169	0.0005	0.0021	0.0015	0.0013	0.0001	0.0021	0.0036
-2.3	0.1166	0.0005	0.0019	0.0018	0.0014	0.0001	0.0021	0.0037
-2.1	0.1166	0.0006	0.0020	0.0020	0.0015	0.0001	0.0020	0.0038
-1.9	0.1191	0.0008	0.0021	0.0019	0.0014	0.0002	0.0016	0.0036
-1.7	0.1173	0.0010	0.0015	0.0023	0.0016	0.0001	0.0019	0.0038
-1.5	0.1175	0.0016	0.0005	0.0029	0.0014	0.0001	0.0017	0.0040
-1.3	0.1159	0.0037	0.0014	0.0029	0.0018	0.0004	0.0011	0.0054

TABLE I: Results of  $\alpha_s(M_Z)$  extracted from the three-jet rate measured by ALEPH at LEP1. The uncertainty contributions are given for the statistical error (stat.), the uncertainty related to the choice of the generator for the simulation of the detector response (det.), the quadratic sum of all other experimental systematic uncertainties arising from track and event selection cut variations (exp.), the hadronisation uncertainty obtained by the maximum difference between either PYTHIA, HERWIG or ARIADNE (had.), the uncertainty on the b-quark mass correction procedure (mass) and the uncertainty for missing higher orders (pert.) estimated by a variation of the renormalisation scale.

- [16] G. Corcella *et al.*, JHEP **0101** (2001) 010.  
[17] L. Lönnblad, Comput. Phys. Commun. **71** (1992) 15.  
[18] C. Anastasiou, K. Melnikov and F. Petriello, Phys. Rev. Lett. **93** (2004) 032002.  
[19] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, Nucl. Phys. B **691** (2004) 195.  
[20] S. Weinzierl, Phys. Rev. D **74** (2006) 014020.  
[21] R.K. Ellis, D.A. Ross and A.E. Terrano, Nucl. Phys. B **178** (1981) 421.  
[22] Z. Kunszt, Phys. Lett. B **99** (1981) 429; J.A.M. Vermaseren, K.J.F. Gaemers and S.J. Oldham, Nucl. Phys. B **187** (1981) 301; K. Fabricius, I. Schmitt, G. Kramer and G. Schierholz, Z. Phys. C **11** (1981) 315.  
[23] Z. Kunszt and P. Nason, in *Z Physics at LEP 1*, CERN Yellow Report 89-08, Vol. 1, p. 373; W. T. Giele and E.W.N. Glover, Phys. Rev. D **46** (1992) 1980; S. Catani and M. H. Seymour, Phys. Lett. B **378** (1996) 287.  
[24] L.J. Dixon and A. Signer, Phys. Rev. Lett. **78** (1997) 811; Phys. Rev. D **56** (1997) 4031.  
[25] Z. Nagy and Z. Trocsanyi, Phys. Rev. Lett. **79** (1997) 3604.  
[26] J. Campbell, M.A. Cullen and E.W.N. Glover, Eur. Phys. J. C **9** (1999) 245.  
[27] S. Weinzierl and D.A. Kosower, Phys. Rev. D **60** (1999) 054028.  
[28] K. Hagiwara and D. Zeppenfeld, Nucl. Phys. B **313** (1989) 560; F.A. Berends, W.T. Giele and H. Kuijf, Nucl. Phys. B **321** (1989) 39; N.K. Falck, D. Graudenz and G. Kramer, Nucl. Phys. B **328** (1989) 317.  
[29] S. Moretti, Nucl. Phys. B **544** (1999) 289.  
[30] F. Krauss, R. Kuhn and G. Soff, JHEP **0202** (2002) 044.  
[31] W. Bernreuther, A. Brandenburg and P. Uwer, Phys. Rev. Lett. **79** (1997) 189; G. Rodrigo, A. Santamaria and M. S. Bilenky, Phys. Rev. Lett. **79** (1997) 193; P. Nason and C. Oleari, Nucl. Phys. B **521** (1998) 237.  
[32] A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover and G. Heinrich, Phys. Rev. Lett. **99** (2007) 132002; JHEP **0711** (2007) 058; JHEP **0712** (2007) 094; JHEP **0905** (2009) 106.  
[33] A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover and G. Heinrich, Phys. Rev. Lett. **100** (2008) 172001.  
[34] S. Weinzierl, Phys. Rev. Lett. **101** (2008) 162001.  
[35] S. Weinzierl, JHEP **0906** (2009) 041; JHEP **0907** (2009) 009; arXiv:0909.5056.  
[36] G. Dissertori *et al.*, JHEP **0802** (2008) 040.  
[37] T. Becher and M.D. Schwartz, JHEP **0807** (2008) 034.  
[38] R.A. Davison and B.R. Webber, Eur. Phys. J. C **59** (2009) 13.  
[39] S. Bethke, S. Kluth, C. Pahl and J. Schieck [JADE Collaboration], arXiv:0810.1389.  
[40] G. Dissertori *et al.*, JHEP **0908** (2009) 036.  
[41] A. Heister *et al.* [ALEPH Collaboration], Eur. Phys. J. C **35** (2004) 457.  
[42] K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, Phys. Lett. B **85** (1979) 277; W. Celmaster and R. J. Goncalves, Phys. Rev. Lett. **44** (1980) 560; M. Dine and J. R. Sapirstein, Phys. Rev. Lett. **43**, 668 (1979); K.G. Chetyrkin, J.H. Kühn and A. Kwiatkowski, Phys. Rept. **277** (1996) 189.  
[43] A. Denner, S. Dittmaier, T. Gehrmann and C. Kurz, Phys. Lett. B **679** (2009) 219.  
[44] A. Banfi, G.P. Salam and G. Zanderighi, JHEP **0201**

- (2002) 018; JHEP **0503** (2005) 073.
- [45] T. Gehrmann, G. Luisoni and H. Stenzel, Phys. Lett. B **664** (2008) 265.
- [46] R. Barate *et al.* [ALEPH Collaboration], Phys. Rep. **294** (1998) 1.